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# Tridiagonal Matrix Algorithm (Thomas Algorithm)

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# 1 TRIDIAGONAL MATRIX ALGORITHM (THOMAS ALGORITHM)

## 1.1 Tridiagonal Linear Systems

A **tridiagonal linear system** is one of the form

$$\boxed{\mathbf{Ax} = \mathbf{d}} \quad (1)$$

where

$$\underbrace{\begin{bmatrix} b_1 & c_1 & & & & & & & & & \\ a_1 & b_2 & c_2 & & & & & & & & \\ & a_2 & \ddots & \ddots & & & & & & & \\ & & \ddots & \ddots & \ddots & & & & & & \\ & & & & & & c_{n-2} & & & & \\ & & & & & & a_{n-2} & b_{n-1} & c_{n-1} & & \\ & & & & & & a_{n-1} & b_n & & & \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{n-1} \\ d_n \end{bmatrix}}_{\mathbf{d}} \quad (2)$$

and where  $\mathbf{A} \in \mathbb{R}^{n \times n}$  and  $\mathbf{x}, \mathbf{d} \in \mathbb{R}^n$ . Owing to the fact that it only has three nonzero diagonals, the matrix  $\mathbf{A}$  is referred to as a **tridiagonal matrix**<sup>1</sup>.

The **tridiagonal matrix algorithm** (also known as the **Thomas algorithm**) is an algorithm that can efficiently solve the tridiagonal linear system for  $\mathbf{x}$ . In this document, we introduce three different ways<sup>2</sup> to implement the tridiagonal matrix algorithm (Algorithms 1, 2, and 3). The first two implementations use three vectors,  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$ , which we define as [1]

$$\mathbf{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_{n-1} \end{bmatrix} \quad (3)$$

## 1.2 Slower Implementation

The tridiagonal matrix algorithm is shown below [1, 3, 4].

<sup>1</sup> In many references, a tridiagonal matrix is often defined with one of the following two convention:

$$\mathbf{A} = \begin{bmatrix} a_1 & b_1 & & & & & & & & & \\ c_1 & a_2 & b_2 & & & & & & & & \\ & c_2 & \ddots & \ddots & & & & & & & \\ & & \ddots & \ddots & \ddots & & & & & & \\ & & & & & & b_{n-2} & & & & \\ & & & & & & c_{n-2} & a_{n-1} & b_{n-1} & & \\ & & & & & & c_{n-1} & a_n & & & \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} b_1 & c_1 & & & & & & & & & \\ a_2 & b_2 & c_2 & & & & & & & & \\ & a_3 & \ddots & \ddots & & & & & & & \\ & & \ddots & \ddots & \ddots & & & & & & \\ & & & & & & c_{n-2} & & & & \\ & & & & & & a_{n-1} & b_{n-1} & c_{n-1} & & \\ & & & & & & a_n & b_n & & & \end{bmatrix}$$

The first is typically used when defining a tridiagonal linear system [2], while the second is used almost exclusively when defining the tridiagonal matrix algorithm [3, 4]. However, for the second convention above, the  $a_i$ 's range from  $a_2$  to  $a_n$ , which is inconvenient from a programming standpoint; therefore, I defined them here as ranging from  $a_1$  to  $a_{n-1}$ . This convention is also reflected in Algorithms 1 and 2.

<sup>2</sup> The `tridiagonal` function implements Algorithm 2.

**Algorithm 1:** tridiagonal\_slower

Tridiagonal matrix algorithm (Thomas algorithm) (slower version).

**Given:**

- $\mathbf{A} \in \mathbb{R}^{n \times n}$  - tridiagonal matrix
- $\mathbf{d} \in \mathbb{R}^n$  - vector

**Note:**

- $\mathbf{A}$  and  $\mathbf{d}$  define the tridiagonal linear system  $\mathbf{Ax} = \mathbf{d}$ .

**Procedure:**

1. Determine  $n$ , given that  $\mathbf{d} \in \mathbb{R}^n$ .
2. Preallocate vectors of size  $n \times 1$  to store  $\mathbf{b}$  and  $\mathbf{x}$ .
3. Preallocate vectors of size  $(n - 1) \times 1$  to store  $\mathbf{a}$  and  $\mathbf{c}$ .
4. Extract  $\mathbf{a}$  from  $\mathbf{A}$ .

```

for  $i = 2$  to  $n$ 
    |    $a_{i-1} = A_{i,i-1}$ 
end

```

5. Extract  $\mathbf{b}$  from  $\mathbf{A}$ .

```

for  $i = 1$  to  $n$ 
    |    $b_i = A_{i,i}$ 
end

```

6. Extract  $\mathbf{c}$  from  $\mathbf{A}$ .

```

for  $i = 2$  to  $n$ 
    |    $c_{i-1} = A_{i-1,i}$ 
end

```

7. Forward elimination.

```

for  $2 = 1$  to  $n$ 
    |    $w = \frac{a_{i-1}}{b_{i-1}}$ 
    |    $b_i = b_i - wc_{i-1}$ 
    |    $d_i = d_i - wd_{i-1}$ 
end

```

8. Backward substitution.

```

 $x_n = \frac{d_n}{b_n}$ 
for  $i = n - 1$  to  $1$  by  $-1$ 
    |    $x_i = \frac{d_i - c_i x_{i+1}}{b_i}$ 
end

```

**Return:**

- $\mathbf{x} \in \mathbb{R}^n$  - solution of the tridiagonal linear system  $\mathbf{Ax} = \mathbf{d}$

### 1.3 Faster Implementation

We can save some computational effort by reducing the number of for loops in Algorithm 1. For smaller systems, this doesn't make a huge impact, but for larger systems, it can halve the time it takes to solve. We can note that four of the loops go "forward" in  $i$ , so we can combine them (with the caveat that we must extract  $b_1$  separately since its loop starts from 1 and not 2). Defining this "faster" algorithm,

**Algorithm 2:** tridiagonal

Tridiagonal matrix algorithm (Thomas algorithm).

**Given:**

- $\mathbf{A} \in \mathbb{R}^{n \times n}$  - tridiagonal matrix
- $\mathbf{d} \in \mathbb{R}^n$  - vector

**Note:**

- $\mathbf{A}$  and  $\mathbf{d}$  define the tridiagonal linear system  $\mathbf{Ax} = \mathbf{d}$ .

**Procedure:**

1. Determine  $n$ , given that  $\mathbf{d} \in \mathbb{R}^n$ .
2. Preallocate vectors of size  $n \times 1$  to store  $\mathbf{b}$  and  $\mathbf{x}$ .
3. Preallocate vectors of size  $(n - 1) \times 1$  to store  $\mathbf{a}$  and  $\mathbf{c}$ .
4. Extract first element of  $\mathbf{b}$  from  $\mathbf{A}$ .

$$b_1 = A_{1,1}$$

5. Forward loop.

**for**  $i = 2$  **to**  $n$

- (a) Extract relevant elements of  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  from  $\mathbf{A}$ .

$$a_{i-1} = A_{i,i-1}$$

$$b_i = A_{i,i}$$

$$c_{i-1} = A_{i-1,i}$$

- (b) Forward elimination.

$$w = \frac{a_{i-1}}{b_{i-1}}$$

$$b_i = b_i - wc_{i-1}$$

$$d_i = d_i - wd_{i-1}$$

**end**

6. Backward loop (backward substitution).

$$x_n = \frac{d_n}{b_n}$$

```

for i = n - 1 to 1 by -1
    |
    |    $x_i = \frac{d_i - c_i x_{i+1}}{b_i}$ 
    |
end

```

**Return:**

- $\mathbf{x} \in \mathbb{R}^n$  - solution of the tridiagonal linear system  $\mathbf{Ax} = \mathbf{d}$

## 1.4 Shortest Implementation

Finally, instead of defining/preallocating the vectors  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$ , we can access the elements of  $\mathbf{A}$  directly. We refer to this as the “shortest” implementation since it results in the least lines of code.

**Algorithm 3:** tridiagonal\_shortest

Tridiagonal matrix algorithm (Thomas algorithm) (shortest implementation).

**Given:**

- $\mathbf{A} \in \mathbb{R}^{n \times n}$  - tridiagonal matrix
- $\mathbf{d} \in \mathbb{R}^n$  - vector

**Note:**

- $\mathbf{A}$  and  $\mathbf{d}$  define the tridiagonal linear system  $\mathbf{Ax} = \mathbf{d}$ .

**Procedure:**

1. Determine  $n$ , given that  $\mathbf{d} \in \mathbb{R}^n$ .
2. Preallocate a vector of size  $n \times 1$  to store  $\mathbf{x}$ .
3. Forward elimination.

```

for i = 2 to n
    |
    |    $w = \frac{A_{i,i-1}}{A_{i-1,i-1}}$ 
    |    $A_{i,i} = A_{i,i} - wA_{i-1,i}$ 
    |    $d_i = d_i - wd_{i-1}$ 
    |
end

```

4. Backward substitution.

```

 $x_n = \frac{d_n}{A_{n,n}}$ 
for i = n - 1 to 1 by -1
    |
    |    $x_i = \frac{d_i - A_{i,i+1}x_{i+1}}{A_{i,i}}$ 
    |
end

```

**Return:**

- $\mathbf{x} \in \mathbb{R}^n$  - solution of the tridiagonal linear system  $\mathbf{Ax} = \mathbf{d}$

For smaller systems, this implementation can actually be the fastest, since you only have to preallocate one vector ( $\mathbf{x}$ ) instead of four ( $\mathbf{x}$ ,  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$ ). However, for large systems, it is costlier to traverse the matrix  $\mathbf{A}$  during the substitution process than it is to preallocate, define, and traverse the vectors  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$ .

## REFERENCES

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