Aerospace Simulation

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PART I

State Parameterization

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Mathematical Vectors

1.1 Mathematical Vectors and Matrices

In mathematics, a **matrix** is a rectangular array of numbers. An example of a matrix is

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 0 & 5 \end{bmatrix} \in \mathbb{R}^{2\times 3}$$

In this text, we refer to such a matrix as a **mathematical matrix**.

Convention 1: Notation for mathematical matrices.

Mathematical matrices are written using uppercase, upright, boldface symbols. For example, a real-valued matrix of size $m \times n$ may be written as

$$\mathbf{A} \in \mathbb{R}^{m\times n}$$

In mathematics, a **vector** is a list of numbers. An example of a vector is

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^3$$

In this text, we refer to such a vector as a **mathematical vector**. Note that it is standard to assume that all vectors are **column vectors**. If a vector is arranged in a row, we refer to it explicitly as a **row vector**, and denote its size differently. For example, an n-dimensional real-valued row vector is denote as being a member of the vector space $\mathbb{R}^{1\times n}$. An example of a row vector is

$$\mathbf{v} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \in \mathbb{R}^{1\times n}$$

Convention 2: Notation for mathematical vectors.

Mathematical vectors are written using lowercase, upright, boldface symbols. For example, a real-valued vector of length n may be written as

$$\mathbf{v} \in \mathbb{R}^n$$

Real-valued mathematical vectors have both a magnitude and a direction. The **magnitude** of mathematical vector $\mathbf{v}$ is calculated using the 2-norm.

$$\boxed{\|\mathbf{v}\| = \text{magnitude of } \mathbf{v}} \tag{1.1}$$

The **2-norm** of a vector is defined as

$$\boxed{\|\mathbf{v}\| = \sqrt{\mathbf{v}^T \mathbf{v}}} \tag{1.2}$$

where the "T" denotes the transpose operation.

To describe the **direction** of a vector, we use unit vectors. A **unit vector** is a vector of magnitude 1. Therefore, If we multiply a scalar quantity by a unit vector, we get a vector whose magnitude is equal to the scalar and whose direction is parallel to the unit vector's. The unit vector in the direction of $\mathbf{v}$ is defined as

$$\boxed{\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \text{direction of } \mathbf{v}} \tag{1.3}$$

Convention 3: Notation for the magnitude of a mathematical vector.

The magnitude of mathematical vectors are written using the italic, non-boldface version of the same symbol. For example, the magnitude of $\mathbf{v}$ is written as v.

By rearranging Eq. (1.3), we can write a vector $\mathbf{v}$ in terms of its magnitude and direction.

$$\boxed{\mathbf{v} = \|\mathbf{v}\| \, \hat{\mathbf{v}}} \tag{1.4}$$

1.2 Coordinate Systems

A **coordinate system** is a system that uses one or more numbers (i.e. coordinates) to determine the position of some geometric element with respect to that coordinate system. We use the following convention to name coordinate systems:

Convention 4: Naming coordinate systems.

Consider a coordinate system with origin O and axes $x_1, ..., x_n$. We refer to this coordinate system as $Ox_1...x_n$.

As a visual example, consider the three-dimensional coordinate system $Ox_1x_2x_3$ is illustrated in Fig. 1.1.

A coordinate system is defined using an **ordered basis**. Let X be the ordered basis defining the $Ox_1...x_n$ coordinate system. Then

$$\boxed{X = (\hat{\mathbf{x}}_1, ..., \hat{\mathbf{x}}_n) = \text{ordered basis}} \tag{1.5}$$

$\hat{\mathbf{x}}_1, ..., \hat{\mathbf{x}}_3$ are referred to as **basis vectors** since they define an ordered basis. Note that all basis vectors are unit vectors, i.e. they all a of have magnitude 1.

$$\boxed{\|\hat{\mathbf{x}}_1\| = \cdots = \|\hat{\mathbf{x}}_n\| = 1} \tag{1.6}$$

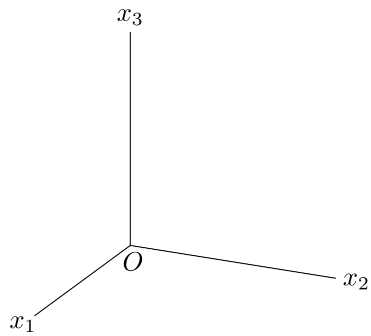


Figure 1.1: $Ox_1x_2x_3$ coordinate system.

Also note that all basis vectors have dimension equal to the dimension of the coordinate system they collectively define. For an n-dimensional coordinate system,

$$\boxed{\hat{\mathbf{x}}_i \in \mathbb{R}^n \quad \forall\, i = 1, ..., n} \tag{1.7}$$

In our three-dimensional example, the tails of $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$, and $\hat{\mathbf{x}}_3$ are all located at the origin O, while their heads points in the directions of the x_1, x_2, and x_3 axes, respectively. The $Ox_1x_2x_3$ coordinate system together, with its basis vectors $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$, and $\hat{\mathbf{x}}_3$, is depicted in Fig. 1.2.

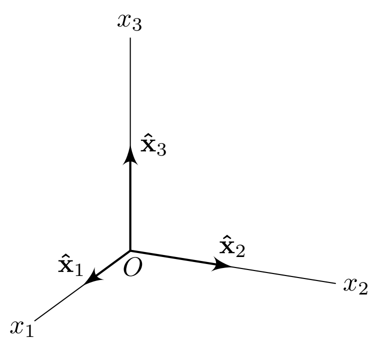


Figure 1.2: $Ox_1x_2x_3$ coordinate system with basis vectors.

1.3 Mathematical Vectors in a Coordinate System

Consider an arbitrary, n-dimensional mathematical vector, $\mathbf{v} \in \mathbb{R}^n$. Let $X = (\mathbf{x}_1, ..., \mathbf{x}_n)$ be an ordered basis defining the $Ox_1...x_n$ coordinate system. Then $\mathbf{v}$ can be written in terms of the basis vectors of the $Ox_1...x_n$ coordinate system as

$$\boxed{\mathbf{v} = \sum_{i=1}^{n} v_{x_i} \hat{\mathbf{x}}_i} \tag{1.8}$$

where v_{x_i} is the **component** or **coordinate** of $\mathbf{v}$ with respect to the x_i axis (i.e. in the direction defined by $\hat{\mathbf{x}}_i$).

As an example, consider a three-dimensional mathematical vector, $\mathbf{v} \in \mathbb{R}^3$. Let's place $\mathbf{v}$ in the three-dimensional coordinate system $Ox_1x_2x_3$ such that its tail is located at the origin. Vector $\mathbf{v}$ has components/coordinates v_1, v_2, and v_3 directed along the three coordinate axes, defined by the basis vectors $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$, and $\hat{\mathbf{x}}_3$, respectively. For this example, $\mathbf{v}$ can be written in terms of the basis vectors defining $Ox_1x_2x_3$ as

$$\mathbf{v} = v_{x_1}\hat{\mathbf{x}}_1 + v_{x_2}\hat{\mathbf{x}}_2 + v_{x_3}\hat{\mathbf{x}}_3$$

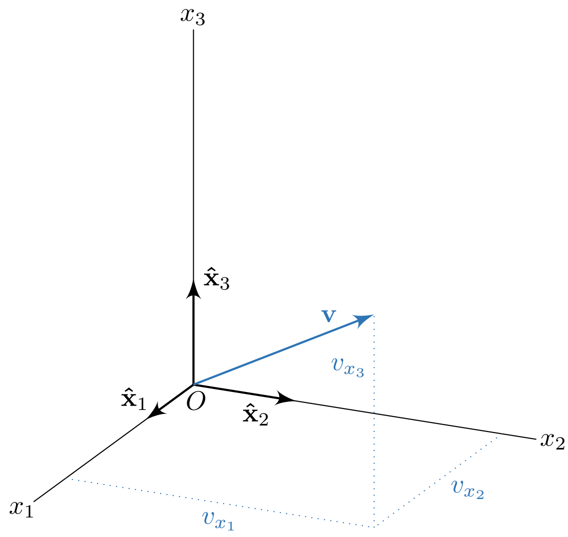


Figure 1.3: Mathematical vector in a coordinate system.

1.4 Labeling a Vector With a Basis

Recall that in the previous section we wrote the vector $\mathbf{v} \in \mathbb{R}^n$ as a linear combination of the basis vectors of the ordered basis $X = (\hat{\mathbf{x}}_1, ..., \hat{\mathbf{x}}_n)$.

$$\mathbf{v} = \sum_{i=1}^{n} v_i \hat{\mathbf{x}}_i$$

However, X may not be the only ordered basis for $\mathbf{R}^n$; in general, *any* set of unit vectors that span $\mathbf{R}^n$ can be an ordered basis for $\mathbf{R}^n$. Since a mathematical vector is at its core just a list of numbers, it is important to label which basis its components are **resolved** or **expressed** in if there are multiple bases. In this case, we would write

$$\boxed{\mathbf{v}_X = \sum_{i=1}^{n} v_{x_i} \hat{\mathbf{x}}_i \qquad \text{for the ordered basis } X = (\hat{\mathbf{x}}_1, ..., \hat{\mathbf{x}}_n)} \tag{1.9}$$

Convention 5: Labeling a vector with a basis.

A mathematical vector defined with respect to an ordered basis should be subscripted with the symbol representing the ordered basis. For example, a mathematical vector defined with respect to the ordered basis X should be denoted

$$\mathbf{v}_X$$

If the vector already has a subscript, the basis symbol should be the last subscript. For example, if a vector already has the subscript "ex", it should be denoted

$$\mathbf{v}_{\text{ex},X}$$

1.4.1 Unit Vectors

Just like regular vectors, we can label unit vectors with a basis as well. This implies that we can define an ordered basis for one coordinate system using basis vectors resolved in another coordinate system.

As an example, consider the ordered basis $X = (\hat{\mathbf{x}}_1^Y, ..., \hat{\mathbf{x}}_n^Y)$. Note that the basis vectors of X are resolved with respect to another ordered basis, Y.

Convention 6: Basis vectors resolved with respect to the basis they define.

Consider an ordered basis $X = (\hat{\mathbf{x}}_{1,X}, ..., \hat{\mathbf{x}}_{n,X})$. To avoid cluttering notation, we write this ordered basis as

$$X = (\hat{\mathbf{x}}_1, ..., \hat{\mathbf{x}}_n)$$

In other words, if a basis is defined using basis vectors resolved in the basis itself, then we do not label the basis vectors with a basis.

$$(\hat{\mathbf{x}}_1, ..., \hat{\mathbf{x}}_n) \equiv (\hat{\mathbf{x}}_{1,X}, ..., \hat{\mathbf{x}}_{n,X})$$

1.5 The Basis Matrix

Consider a coordinate system $Ox_1...x_n$ defined using the ordered basis $X = (\hat{\mathbf{x}}_{1,U}, ..., \hat{\mathbf{x}}_{n,U})$. Note that the basis vectors of X are resolved with respect to some **universal basis**, $U = (\hat{\mathbf{u}}_1, ..., \hat{\mathbf{u}}_n)$. A vector resolved in the coordinate system defined by the basis U can be written as a linear combination of the basis vectors of X as

$$\mathbf{v}_U = \sum_{i=1}^{n} v_{x_i} \hat{\mathbf{x}}_{i,U} = v_{x_1} \hat{\mathbf{x}}_{1,U} + \cdots + v_{x_n} \hat{\mathbf{x}}_{n,U} \tag{1.10}$$

Note that $\hat{\mathbf{x}}_{i,U}$ is essentially $\hat{\mathbf{x}}_i = \hat{\mathbf{x}}_i^X$ projected onto $\hat{\mathbf{u}}_i$. Thus, each scalar x_i represents a coordinate on the ith axis of the coordinate system defined by the ordered basis U.

Equivalently, we can write Eq. (1.10) using matrix algebra as

$$\mathbf{v}_U = \begin{bmatrix} \hat{\mathbf{x}}_{1,U} & \cdots & \hat{\mathbf{x}}_{n,U} \end{bmatrix} \begin{bmatrix} v_{x_1} \\ \vdots \\ v_{x_n} \end{bmatrix} \tag{1.11}$$

Let's define the **basis matrix** of X with respect to U as

$$\boxed{\mathbf{X}_U = \begin{bmatrix} \hat{\mathbf{x}}_{1,U} & \cdots & \hat{\mathbf{x}}_{n,U} \end{bmatrix}} \tag{1.12}$$

Note that since $\hat{\mathbf{x}}_i \in \mathbb{R}^n \; \forall \; i = 1, ..., n$, we have that

$$\boxed{\mathbf{X}_U \in \mathbb{R}^{n \times n}} \tag{1.13}$$

1.5.1 The Standard Basis Matrix

Consider the case where the basis vectors of an ordered basis are defined with respect to the basis itself. Mathematically, if we have n-dimensional ordered basis X, then

$$X = (\hat{\mathbf{x}}_1, ..., \hat{\mathbf{x}}_n)$$

It is important to recall Convention 6, which notes that this notation is shorthand for $X = (\hat{\mathbf{x}}_{1,X}, ..., \hat{\mathbf{x}}_{n,X})$.

A vector resolved in the coordinate system defined by the basis X can be written in terms of the basis vectors of X as

$$\mathbf{v}_X = v_{x_1} \hat{\mathbf{x}}_1 + \cdots + v_{x_n} \hat{\mathbf{x}}_n$$

Equivalently, we can write this using matrix algebra as

$$\mathbf{v}_X = \begin{bmatrix}\hat{\mathbf{x}}_1 & \cdots & \hat{\mathbf{x}}_n\end{bmatrix}\begin{bmatrix}v_{x_1}\\ \vdots \\ v_{x_n}\end{bmatrix}$$

Let's define the **standard basis matrix** of a basis with respect to itself as

$$\mathbf{X}_X = \begin{bmatrix}\hat{\mathbf{x}}_1 & \cdots & \hat{\mathbf{x}}_n\end{bmatrix}$$

Using the standard basis matrix $\mathbf{X}$, we can write $\mathbf{v}_X$ as

$$\mathbf{v}_X = \mathbf{X}\begin{bmatrix}v_{x_1}\\ \vdots \\ v_{x_n}\end{bmatrix}$$

By definition, we know that $(v_{x_1}, ..., v_{x_n})^T = \mathbf{v}_X$. Thus, we have

$$\mathbf{v}_X = \mathbf{X}_X \mathbf{v}_X$$

which implies that the standard basis matrix is just the identity matrix, $\mathbf{I}_{n\times n}$.

$$\boxed{\mathbf{X}_X = \mathbf{I}_{n\times n} \quad \text{for the ordered basis } X = (\hat{\mathbf{x}}_1, ..., \hat{\mathbf{x}}_n)} \tag{1.14}$$

1.6 Change of Basis

The developments in this section are compiled from [4, pp. 163–172] and [7, pp. 33–34].

Consider two separate ordered basis, X and Y.

$$X = (\hat{\mathbf{x}}_1, ..., \hat{\mathbf{x}}_n)$$
$$Y = (\hat{\mathbf{y}}_1, ..., \hat{\mathbf{y}}_n)$$

Supposed the tails of all the basis vectors are all located at the same point, O. They then define the coordinate systems $Ox_1...x_n$ and $Oy_1...y_n$, both with origin O.

Goals:

1. Resolve $\mathbf{v}$ in the $Oy_1...y_n$ coordinate system, given its coordinates in the $Ox_1...x_n$ coordinate system. This operation represents the **change of basis** from X to Y; we obtain $\mathbf{v}_Y$ given $\mathbf{v}_X$.
2. Resolve $\mathbf{v}$ in the $Ox_1...x_n$ coordinate system, given its coordinates in the $Oy_1...y_n$ coordinate system. This operation represents the **change of basis** from Y to X; we obtain $\mathbf{v}_X$ given $\mathbf{v}_Y$.

1.6.1 Generalized Change of Basis

Recall Eq. (1.11) defining $\mathbf{v}_U$ in terms of the basis matrix of X with respect to U, where $X = (\hat{\mathbf{x}}_{1,U}, ..., \hat{\mathbf{x}}_{n,U})$ is some ordered basis and $U = (\hat{\mathbf{u}}_1, ..., \hat{\mathbf{u}}_n)$ is a universal ordered basis.

$$\mathbf{v}_U = \underbrace{\begin{bmatrix}\hat{\mathbf{x}}_{1,U} & \cdots & \hat{\mathbf{x}}_{n,U}\end{bmatrix}}_{\mathbf{X}_U \text{ (Eq. (1.12))}}\underbrace{\begin{bmatrix}v_{x_1}\\ \vdots \\ v_{x_n}\end{bmatrix}}_{\mathbf{v}_X}$$

As noted in the equation above, the first matrix is just the basis matrix of X with respect to U, while the vector is the vector $\mathbf{v}$ resolved with respect to the ordered basis X. Thus, we have

$$\mathbf{v}_U = \mathbf{X}_U \mathbf{v}_X \tag{1.15}$$

Similarly, we could write $\mathbf{v}_U$ in terms of $\mathbf{v}_Y$ and $\mathbf{Y}_U$, where $Y = (\hat{\mathbf{y}}_{1,U}, ..., \hat{\mathbf{y}}_{n,U})$ is another ordered basis with basis vectors resolved with respect to the universal ordered basis U.

$$\mathbf{v}_U = \mathbf{Y}_U \mathbf{v}_Y \tag{1.16}$$

From Eqs. (1.15) and (1.16), we have

$$\mathbf{X}_U \mathbf{v}_X = \mathbf{Y}_U \mathbf{v}_Y \tag{1.17}$$

Solving for $\mathbf{v}_Y$ by left-multiplying both sides by $\mathbf{Y}_U^{-1}$,

$$\mathbf{v}_Y = \mathbf{Y}_U^{-1} \mathbf{X}_U \mathbf{v}_X$$

Below, we more formally define this change of basis.

Formal Definitions

Consider two separate ordered basis, X and Y, which both have basis vectors resolved with respect to a third, universal, ordered basis U.

$$X = (\hat{\mathbf{x}}_{1,U}, ..., \hat{\mathbf{x}}_{n,U})$$
$$Y = (\hat{\mathbf{y}}_{1,U}, ..., \hat{\mathbf{y}}_{n,U})$$
$$U = (\hat{\mathbf{u}}_1, ..., \hat{\mathbf{u}}_n)$$

Suppose we know the basis vectors of X and Y resolved with respect to U.

$$\mathbf{X}_U = \begin{bmatrix} \hat{\mathbf{x}}_{1,U} & \cdots & \hat{\mathbf{x}}_{n,U} \end{bmatrix}$$

$$\mathbf{Y}_U = \begin{bmatrix} \hat{\mathbf{y}}_{1,U} & \cdots & \hat{\mathbf{y}}_{n,U} \end{bmatrix}$$

The **generalized change of basis** from X to Y is defined as

$$\boxed{\mathbf{v}_Y = \mathbf{Y}_U^{-1} \mathbf{X}_U \mathbf{v}_X \qquad \text{(generalized change of basis } X \to Y\text{)}} \tag{1.18}$$

Similarly, the **generalized change of basis** from Y to X is defined as

$$\boxed{\mathbf{v}_X = \mathbf{X}_U^{-1} \mathbf{Y}_U \mathbf{v}_Y \qquad \text{(generalized change of basis } Y \to X\text{)}} \tag{1.19}$$

1.6.2 Standard Change of Basis

From Eq. (1.18), we know that the generalized change of basis from X to Y is

$$\mathbf{v}_Y = (\mathbf{Y}_U)^{-1} \mathbf{X}_U \mathbf{v}_X$$

where $U = (\hat{\mathbf{u}}_1, ..., \hat{\mathbf{u}}_n)$ is some universal basis that both X and Y are defined with respect to. Now, let's consider the case where $U = Y$, which essentially means that we have the basis vectors of X resolved with respect to the basis Y.

$$\mathbf{v}_Y = \underbrace{\mathbf{Y}_Y^{-1}}_{\mathbf{I}_{n\times n}} \mathbf{X}_Y \mathbf{v}_X$$

Note that $\mathbf{Y}_Y$ is a standard basis matrix, which we showed in Section 1.5.1 is just the identity matrix. Therefore, this change of basis reduces to

$$\mathbf{v}_Y = \mathbf{X}_Y \mathbf{v}_X$$

We can perform the same procedure with Eq. (1.19), setting $U = X$, to find

$$\mathbf{v}_X = \mathbf{Y}_X \mathbf{v}_Y$$

Below, we more formally define this change of basis.

Formal Definitions

Consider two separate ordered basis, X and Y. Assume that we have the vector $\mathbf{v}_X$ resolved with respect to X, together with the basis vectors of Y resolved with respect to X. The basis vectors of Y form the basis matrix

$$\mathbf{Y}_X = \begin{bmatrix} \hat{\mathbf{y}}_{1,X} & \cdots & \hat{\mathbf{y}}_{n,X} \end{bmatrix}$$

The **standard change of basis** from X to Y is defined as

$$\boxed{\mathbf{v}_Y = \mathbf{X}_Y \mathbf{v}_X \qquad \text{(standard change of basis } X \to Y\text{)}} \tag{1.20}$$

Similarly, assume that we have the vector $\mathbf{v}_Y$ resolved with respect to Y, together with the basis vectors of X resolved with respect to Y. The basis vectors of X form the basis matrix

$$\mathbf{X}_Y = \begin{bmatrix} \hat{\mathbf{x}}_{1,Y} & \cdots & \hat{\mathbf{x}}_{n,Y} \end{bmatrix}$$

The **standard change of basis** from Y to X is defined as

$$\boxed{\mathbf{v}_X = \mathbf{Y}_X \mathbf{v}_Y \qquad \text{(standard change of basis } Y \to X\text{)}} \tag{1.21}$$

Note that we can also obtain $\mathbf{v}_X$ by left-multiplying both sides of Eq. (1.20) by $\mathbf{X}_Y^{-1}$.

$$\mathbf{v}_Y = \mathbf{X}_Y \mathbf{v}_X \quad \rightarrow \quad \mathbf{X}_Y^{-1} \mathbf{v}_Y = \mathbf{X}_Y^{-1} \mathbf{X}_Y \mathbf{v}_X \quad \rightarrow \quad \mathbf{X}_Y^{-1} \mathbf{v}_Y = \mathbf{v}_X$$

$$\therefore \mathbf{v}_X = \mathbf{X}_Y^{-1} \mathbf{v}_Y$$

Comparing this result with Eq. (1.21) implies that

$$\boxed{\mathbf{Y}_X = \mathbf{X}_Y^{-1}} \tag{1.22}$$

Repeating the same process but this time left-multiplying both sides of Eq. (1.21) by $\mathbf{Y}_X^{-1}$ and then comparing the result with Eq. (1.20) would result in

$$\boxed{\mathbf{X}_Y = \mathbf{Y}_X^{-1}} \tag{1.23}$$

1.7 Orthogonality

1.7.1 Orthogonal Matrices

Orthogonal Vectors

Two vectors, $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^n$, are said to be orthogonal if their inner product is 0.

$$\boxed{\mathbf{v}_1^T \mathbf{v}_2 = 0 \quad \rightarrow \quad \mathbf{v}_1 \in \mathbb{R}^n \text{ and } \mathbf{v}_2 \in \mathbb{R}^n \text{ are orthogonal}} \tag{1.24}$$

Orthonormal Sets

An **orthonormal set** is a set of unit vectors that are all orthogonal to one another. Let $U = \{\hat{\mathbf{u}}_1, ..., \hat{\mathbf{u}}_n\}$ be a set of n vectors such that

$$\hat{\mathbf{u}}_i^T \hat{\mathbf{u}}_j = \delta_{ij}$$

where

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Then U is an orthonormal set.

Note that the condition above implies that all the vectors are both orthogonal to one another and have magnitude 1. For the case where $i \neq j$, $\hat{\mathbf{u}}_i^T \hat{\mathbf{u}}_j = 0$ implies that the vectors are orthogonal. For the case where $i = j$, $\hat{\mathbf{u}}_i^T \hat{\mathbf{u}}_i = 1$ implies that $\hat{\mathbf{u}}_i$ is a unit vector, since $\hat{\mathbf{u}}_i^T \hat{\mathbf{u}}_i = \|\hat{\mathbf{u}}_i\|^2$.

Orthogonal Matrices

A square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is said to be **orthogonal** if its column vectors form an orthonormal set. Let $\mathbf{a}_i$ be the ith column vector of $\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_i & \cdots & \mathbf{a}_n \end{bmatrix}$$

$\mathbf{A}$ is an orthogonal matrix if

$$\|\mathbf{a}_i\| = 1 \; \forall \; i = 1, .., n$$
$$\mathbf{a}_i^T \mathbf{a}_j = 0 \; \forall \; i, j = 1, .., n, \; i \neq j$$

Properties of Orthogonal Matrices

Consider multiplying $\mathbf{A}$ by its transpose from the left. Writing the matrices in terms of their column vectors,

$$\mathbf{A}^T \mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}^T \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_n^T \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^T \mathbf{a}_1 & \mathbf{a}_1^T \mathbf{a}_2 & \cdots & \mathbf{a}_1^T \mathbf{a}_n \\ \mathbf{a}_2^T \mathbf{a}_1 & \mathbf{a}_2^T \mathbf{a}_2 & \cdots & \mathbf{a}_2^T \mathbf{a}_n \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_n^T \mathbf{a}_1 & \mathbf{a}_n^T \mathbf{a}_2 & \cdots & \mathbf{a}_n^T \mathbf{a}_n \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

It follows that an orthogonal matrix has the property

$$\boxed{\mathbf{A}^T \mathbf{A} = \mathbf{A}\mathbf{A}^T = \mathbf{I}_{n \times n}} \tag{1.25}$$

where $\mathbf{I}_{n \times n}$ is the $n \times n$ identity matrix.

We also know that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_{n \times n}$, which when compared to Eq. (1.25) yields the property

$$\boxed{\mathbf{A}^{-1} = \mathbf{A}^T} \tag{1.26}$$

Eq. (1.26) is an extremely important property because it is computationally much more efficient to calculate the transpose of a matrix than its inverse.

Next, consider the determinant of $\mathbf{A}\mathbf{A}^T$. We know that

$$\det(\mathbf{I}_{n \times n}) = 1$$

By the definition of the determinant, we know that

$$\det\left(\mathbf{A}\mathbf{A}^T\right) = \det\left(\mathbf{A}\right)\det\left(\mathbf{A}^T\right)$$

Since $\det\left(\mathbf{A}^T\right) = \det\left(\mathbf{A}\right)$ for any square matrix $\mathbf{A}$,

$$\det\left(\mathbf{A}\mathbf{A}^T\right) = \det\left(\mathbf{A}\right)^2$$

Since $\mathbf{A}\mathbf{A}^T = \mathbf{I}_{n\times n}$,

$$\det\left(\mathbf{I}_{n\times n}\right) = \det\left(\mathbf{A}\right)^2 \quad \rightarrow \quad \det\left(\mathbf{A}\right)^2 = 1$$

$$\boxed{\left|\det\left(\mathbf{A}\right)\right| = 1} \tag{1.27}$$

In Eq. (1.27), $|\cdot|$ denotes an absolute value while $\det(\cdot)$ denotes a determinant; this is essential to note because $|\cdot|$ is often used to denote the determinant [5].

> If a matrix is orthogonal, its determinant is either -1 or 1. ***However***, if the determinant of a square matrix is -1 or 1, it does not imply that the matrix is orthogonal [2].

Now, consider two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and the orthogonal matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$. Taking the inner product of $\mathbf{Ax}$ with $\mathbf{Ay}$,

$$(\mathbf{Ax})^T(\mathbf{Ay}) = \mathbf{x}^T \underbrace{\mathbf{A}^T\mathbf{A}}_{\mathbf{I}_{n\times n}} \mathbf{y}$$

$$\boxed{(\mathbf{Ax})^T(\mathbf{Ay}) = \mathbf{x}^T\mathbf{y}} \tag{1.28}$$

In the case that $\mathbf{x} = \mathbf{y}$,

$$(\mathbf{Ax})^T\mathbf{Ax} = \mathbf{x}^T\mathbf{x} = \|\mathbf{x}\|_2^2$$

We also know that

$$(\mathbf{Ax})^T(\mathbf{Ax}) = \|\mathbf{Ax}\|^2$$

Thus, we have [4, pp. 263–268][8, p. 15]

$$\boxed{\|\mathbf{Ax}\| = \|\mathbf{x}\|} \tag{1.29}$$

1.7.2 Product of Orthogonal Matrices

Consider two orthogonal matrices $\mathbf{A}$ and $\mathbf{B}$. Their product is given by $\mathbf{AB}$. Thus, from Eq. (1.26), we know that if $(\mathbf{AB})^T(\mathbf{AB}) = \mathbf{I}$, then the matrix product $\mathbf{AB}$ is also orthogonal.

$$(\mathbf{AB})^T(\mathbf{AB}) = \left(\mathbf{B}^T\mathbf{A}^T\right)(\mathbf{AB}) = \mathbf{B}^T\left(\mathbf{A}^T\mathbf{A}\right)\mathbf{B} = \mathbf{B}^T\mathbf{I}\mathbf{B} = \mathbf{B}^T\mathbf{B} = \mathbf{I}$$

Thus, the product of two orthogonal matrices is also an orthogonal matrix [11].

1.7.3 Orthonormal Bases and Cartesian Coordinate Systems

Consider an ordered basis $X = (\hat{\mathbf{x}}_1, ..., \hat{\mathbf{x}}_n)$ with a corresponding basis matrix

$$\mathbf{X} = \begin{bmatrix} \hat{\mathbf{x}}_1 & \cdots & \hat{\mathbf{x}}_n \end{bmatrix}$$

If $\mathbf{X}$ is an orthogonal matrix, or equivalently, if all the basis vectors defining X are mutually orthogonal, then X is an ordered basis for a **Cartesian coordinate system**.

Let's consider the case where X and Y are both orthonormal ordered bases defining coordinate systems. Then the generalized change of basis formulas from Section 1.6.1 can be simplified to

$$\boxed{\mathbf{v}_Y = \mathbf{Y}_U^T\mathbf{X}_U\mathbf{v}_X \qquad \text{(generalized change of basis } X \rightarrow Y \text{ where } Y \text{ is an orthonormal basis)}} \tag{1.30}$$

$$\boxed{\mathbf{v}_X = \mathbf{X}_U^T \mathbf{Y}_U \mathbf{v}_Y \qquad \text{(generalized change of basis } Y \to X \text{ where } X \text{ is an orthonormal basis)}} \tag{1.31}$$

Additionally, we know that $\mathbf{X}_Y$ and $\mathbf{Y}_X$ are both orthogonal matrices since the basis vectors that comprise them form orthonormal sets. Thus, we can replace the inverses in Eqs. (1.22) and (1.23) to obtain

$$\boxed{\mathbf{Y}_X = \mathbf{X}_Y^T \quad \text{(if } X \text{ is an orthonormal basis)}} \tag{1.32}$$

$$\boxed{\mathbf{X}_Y = \mathbf{Y}_X^T \quad \text{(if } Y \text{ is an orthonormal basis)}} \tag{1.33}$$

Note that these definitions are much faster computationally since transposing a matrix requires much fewer operations than performing a matrix inversion.

1.8 Rotation Matrices

Recall from Section 1.7.3 that a Cartesian coordinate system is defined by an orthonormal basis. While we already simplified the generalized change of basis for Cartesian coordinate systems, there are special ways we can perform a change of basis in a three-dimensional Euclidian space.

In a three-dimensional Euclidian space, the change of basis from one Cartesian coordinate system to another can be represented using rotations.

1.8.1 Passive vs. Active Rotations

When dealing with rotations of coordinate spaces and vectors, there are two common ways that rotations are performed:

1. **Passive Rotation:** The original coordinate system is rotated by some angle θ about one of its axes, while any vector or matrix quantities remain constant, i.e. *passive*, with respect to the original coordinate systems. The vector and matrix quantities are then be resolved in the new coordinate system.
2. **Active Rotation:** The coordinate system remains stationary and the vector or matrix *actively* rotates with respect to the original coordinate system.

In fields such as computer graphics and video games, active rotations are generally used because there is usually some object moving in a fixed coordinate system. However, in the field of dynamics, specifically aerospace dynamics, we typically use passive rotations, since we will have a vector that we want to *resolve* or *express* in different coordinate systems [1].

1.8.2 Elementary Rotations

Consider the $Ox_1x_2x_3$ coordinate system. We refer to the three axes of the this coordinate system as either the **first**, **second**, or **third axes**.

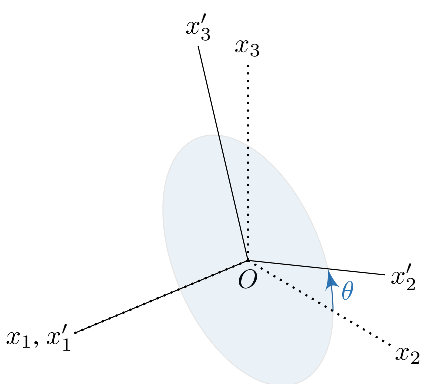
Convention 7: Numbering axes.

1. $x_1 =$ 1st axis
2. $x_2 =$ 2nd axis
3. $x_3 =$ 3rd axis

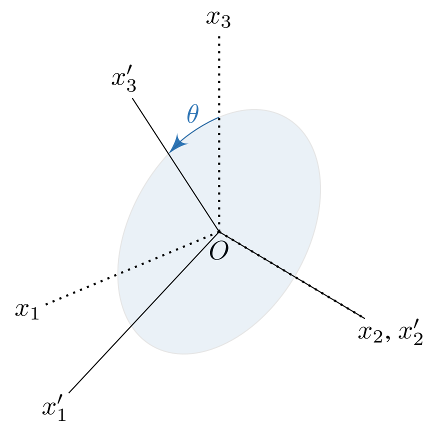
We define the **rotation matrix** for a counterclockwise rotation of θ about the ith axis of a coordinate system as $\mathbf{R}_i(\theta)$. There are three **elementary rotations**, each about a different axes, encoded by $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}_3$, respectively:

1. $\mathbf{R}_1(\theta)$: Encodes counterclockwise rotation by θ of the 2nd and 3rd axes (x_2x_3-plane) about the 1st axis (x_1).
2. $\mathbf{R}_2(\theta)$: Encodes counterclockwise rotation by θ of the 1st and 3rd axes (x_1x_3-plane) about the 2nd axis (x_2).
3. $\mathbf{R}_3(\theta)$: Encodes counterclockwise rotation by θ of the 1st and 2nd axes (x_1x_2-plane) about the 3rd axis (x_3).

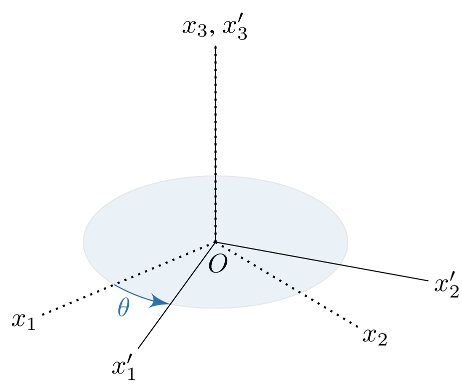
These rotations are illustrated in Fig. 1.4. The rotation matrices $\mathbf{R}_1(\theta)$, $\mathbf{R}_2(\theta)$, and $\mathbf{R}_3(\theta)$ are defined by Eqs. (1.34),



(a) Rotation about 1st axis.



(b) Rotation about 2nd axis.



(c) Rotation about 3rd axis.

Figure 1.4: Elementary rotations.

(1.35), and (1.36), respectively [9, p. 162], [3], [6].

$$\boxed{\mathbf{R}_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}} \tag{1.34}$$

$$\boxed{\mathbf{R}_2(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}} \tag{1.35}$$

$$\boxed{\mathbf{R}_3(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}} \tag{1.36}$$

We formalize these simple equations as algorithms since they will be used often in other algorithms.

Algorithm 1: `rot1`
Rotation matrix for a passive rotation about the 1st axis.

Inputs:
- $\theta \in \mathbb{R}$ - angle of rotation $[\mathrm{rad}]$

Procedure:
1. Precompute the trigonometric functions.
$$c = \cos\theta$$
$$s = \sin\theta$$
2. Construct the rotation matrix.
$$\mathbf{R}_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{bmatrix}$$

Outputs:
- $\mathbf{R}_1(\theta) \in \mathbb{R}^{3\times 3}$ - rotation matrix about 1st axis (passive)

Algorithm 2: `rot2`
Rotation matrix for a passive rotation about the 2nd axis.

Inputs:
- $\theta \in \mathbb{R}$ - angle of rotation $[\mathrm{rad}]$

Procedure:
1. Precompute the trigonometric functions.
$$c = \cos\theta$$
$$s = \sin\theta$$
2. Construct the rotation matrix.
$$\mathbf{R}_2(\theta) = \begin{bmatrix} c & 0 & -s \\ 0 & 1 & 0 \\ s & 0 & c \end{bmatrix}$$

Outputs:
- $\mathbf{R}_2(\theta) \in \mathbb{R}^{3\times 3}$ - rotation matrix about 2nd axis (passive)

Algorithm 3: `rot3`
Rotation matrix for a passive rotation about the 3rd axis.

Inputs:
- $\theta \in \mathbb{R}$ - angle of rotation $[\mathrm{rad}]$

Procedure:

1. Precompute the trigonometric functions.

$$c = \cos\theta$$
$$s = \sin\theta$$

2. Construct the rotation matrix.

$$\mathbf{R}_3(\theta) = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Outputs:

- $\mathbf{R}_3(\theta) \in \mathbb{R}^{3\times3}$ - rotation matrix about 3rd axis (passive)

1.8.3 Properties of Elementary Rotation Matrices

Let $X = (\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \hat{\mathbf{x}}_3)$ and $X' = (\hat{\mathbf{x}}_1', \hat{\mathbf{x}}_2', \hat{\mathbf{x}}_3)$ be two ordered bases that are both orthonormal bases, where X' is defined by rotating X about the x_3 axis by an angle θ. Consider a vector, $\mathbf{v}_X$, resolved in the coordinate system defined by X. To resolve that same vector in the coordinate system defined by X', we could perform the change of basis as defined by Eq. (1.20).

$$\mathbf{v}_{X'} = \mathbf{X}_{X'}\mathbf{v}_X$$

However, we can note that this same change of basis can be performed by the elementary rotation matrix $\mathbf{R}_3(\theta)$.

$$\mathbf{v}_{X'} = \mathbf{R}_3(\theta)\mathbf{v}_X$$

Since $\mathbf{X}_{X'}$ is orthogonal (X is an orthonormal basis), and since $\mathbf{R}_3(\theta) = \mathbf{X}_{X'}$, we know that $\mathbf{R}_3(\theta)$ is orthogonal. In general,

$$\boxed{\text{Elementary rotation matrices, } \mathbf{R}_i(\theta)\text{, are orthogonal matrices.}}$$

Consider defining an elementary rotation matrix using a negative angle.

$$\mathbf{R}_1(-\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(-\theta) & \sin(-\theta) \\ 0 & -\sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}^T = \mathbf{R}_1(\theta)^T$$

$$\mathbf{R}_2(-\theta) = \begin{bmatrix} \cos(-\theta) & 0 & -\sin(-\theta) \\ 0 & 1 & 0 \\ \sin(-\theta) & 0 & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}^T = \mathbf{R}_2(\theta)^T$$

$$\mathbf{R}_3(-\theta) = \begin{bmatrix} \cos(-\theta) & \sin(-\theta) & 0 \\ -\sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}^T = \mathbf{R}_3(\theta)^T$$

Thus, in general, we have

$$\mathbf{R}_i(-\theta) = \mathbf{R}_i(\theta)^T$$

However, we also know that since $\mathbf{R}_i(\theta)$ is orthogonal, its inverse is equal to its transpose. Therefore, we have

$$\boxed{\mathbf{R}_i(\theta)^{-1} = \mathbf{R}_i(\theta)^T = \mathbf{R}_i(-\theta)} \tag{1.37}$$

Since elementary rotation matrices are orthogonal, they also share all the other properties of orthogonal matrices as summarized in Section 1.7.1. Note that in the properties below, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$.

$$\boxed{|\det[\mathbf{R}_i(\theta)]| = 1} \tag{1.38}$$

$$\boxed{\mathbf{R}_i(\theta)^T\mathbf{R}_i(\theta) = \mathbf{R}_i(\theta)\mathbf{R}_i(\theta)^T = \mathbf{I}_{3\times 3}} \tag{1.39}$$

$$\boxed{\left[\mathbf{R}_i(\theta)\mathbf{x}\right]^T\left[\mathbf{R}_i(\theta)\mathbf{y}\right] = \mathbf{x}^T\mathbf{y}} \tag{1.40}$$

$$\boxed{\|\mathbf{R}_i(\theta)\mathbf{x}\| = \|\mathbf{x}\|} \tag{1.41}$$

Since $\mathbf{R}_i(-\theta) = \mathbf{R}_i(\theta)^T$, we can also get an alternate version of the property presented by Eq. (1.39).

$$\boxed{\mathbf{R}_i(\theta)\mathbf{R}_i(-\theta) = \mathbf{I}_{3\times 3}} \tag{1.42}$$

1.8.4 Sequential Rotations

Consider an arbitrary vector $\mathbf{v} \in \mathbb{R}^3$ resolved with respect to the ordered basis $S = (\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$, where S defines the coordinate system $Oxyz$.

$$\mathbf{v}_S = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

First, rotating S by θ_1 about the 1st axis (x), we obtain a new coordinate system $Ox'y'z'$ defined by the ordered basis $S' = (\hat{\mathbf{x}}', \hat{\mathbf{y}}', \hat{\mathbf{z}}')$. Resolving $\mathbf{v}$ in the $Ox'y'z'$ coordinate system (i.e. with respect to the ordered basis S'),

$$\mathbf{v}_{S'} = \mathbf{R}_1(\theta_1)\mathbf{v}_S$$

Next, rotating S' by θ_2 about the 2nd axis (y'), we obtain a third coordinate system, $Ox''y''z''$ defined by the ordered basis $S'' = (\hat{\mathbf{x}}'', \hat{\mathbf{y}}'', \hat{\mathbf{z}}'')$. Resolving $\mathbf{v}$ in the Ox"y"z" coordinate system (i.e. with respect to the ordered basis S''),

$$\begin{aligned} \mathbf{v}_{S''} &= \mathbf{R}_2(\theta_2)\mathbf{v}_{S'} \\ &= \mathbf{R}_2(\theta_2)\mathbf{R}_1(\theta_1)\mathbf{v}_S \end{aligned}$$

In general, for a rotation sequence $i \to j \to k$ (where i, j, and k can be either 1, 2, or 3, i.e. representing an axis), we define

$$\boxed{\mathbf{R}_{ijk}(\theta_1, \theta_2, \theta_3) = \mathbf{R}_k(\theta_3)\mathbf{R}_j(\theta_2)\mathbf{R}_i(\theta_1) \qquad \text{(for the rotation sequence } i \to j \to k\text{)}} \tag{1.43}$$

Convention 8: Sequential rotation.

The ijk rotation sequence ($i \to j \to k$) is described by the rotation matrix

$$\mathbf{R}_{ijk}(\theta_1, \theta_2, \theta_3)$$

where

1. The angles are input in the order the rotations are applied, so their subscript corresponds to the current rotation. For example, θ_2 is used for the second rotation.
2. All angles assumes counterclockwise is positive.
3. The rotations are applied in the order of the subscript:
   (a) First rotation is about the ith axis.
   (b) Second rotation is about the jth axis.
   (c) Third rotation is about the kth axis.

There are two specific rotation sequences that are typically used in aerospace simulation. The two sequences are described in detail below.

3-2-1 Rotation sequence

The 3-2-1 ($3 \to 2 \to 1$) Rotation sequence consists of the following steps:

1. rotation of θ_1 about z (3rd axis)
2. rotation of θ_2 about y' (2nd axis)
3. rotation of θ_3 about x'' (1st axis)

The rotation matrix for the 3-2-1 sequence is [3][1], [10, pp. 763-764][2]

$$\begin{aligned}\mathbf{R}_{321}(\theta_1,\theta_2,\theta_3) &= \mathbf{R}_1(\theta_3)\mathbf{R}_2(\theta_2)\mathbf{R}_3(\theta_1)\\ &= \begin{bmatrix}\cos\theta_3 & \sin\theta_3 & 0\\ -\sin\theta_3 & \cos\theta_3 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\theta_2 & 0 & -\sin\theta_2\\ 0 & 1 & 0\\ \sin\theta_2 & 0 & \cos\theta_2\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta_1 & \sin\theta_1\\ 0 & -\sin\theta_1 & \cos\theta_1\end{bmatrix}\end{aligned}$$

$$\boxed{\mathbf{R}_{321}(\theta_1,\theta_2,\theta_3) = \begin{bmatrix}\cos\theta_2\cos\theta_1 & \cos\theta_2\sin\theta_1 & -\sin\theta_2\\ -\cos\theta_3\sin\theta_1+\sin\theta_3\sin\theta_2\cos\theta_1 & \cos\theta_3\cos\theta_1+\sin\theta_3\sin\theta_2\sin\theta_1 & \sin\theta_3\cos\theta_2\\ \sin\theta_3\sin\theta_1+\cos\theta_3\sin\theta_2\cos\theta_1 & -\sin\theta_3\cos\theta_1+\cos\theta_3\sin\theta_2\sin\theta_1 & \cos\theta_3\cos\theta_2\end{bmatrix}} \tag{1.44}$$

We formalize Eq. (1.44) as Algorithm 4 below. Note that in its implementation, we precompute the trigonometric functions *before* populating the matrix to decrease the computational cost.

Algorithm 4: `rot321`
Rotation matrix for the 3-2-1 rotation sequence.

Inputs:
- $\theta_1 \in \mathbb{R}$ - angle for first rotation (about 3rd axis) [rad]
- $\theta_2 \in \mathbb{R}$ - angle for second rotation (about 2nd axis) [rad]
- $\theta_3 \in \mathbb{R}$ - angle for third rotation (about 1st axis) [rad]

Procedure:
1. Precompute the trigonometric functions.

$$\begin{aligned}s_1 &= \sin\theta_1\\ c_1 &= \cos\theta_1\\ s_2 &= \sin\theta_2\\ c_2 &= \cos\theta_2\\ s_3 &= \sin\theta_3\\ c_3 &= \cos\theta_3\end{aligned}$$

2. Construct the rotation matrix.

$$\mathbf{R}_{321}(\theta_1,\theta_2,\theta_3) = \begin{bmatrix}c_2c_1 & c_2s_1 & -s_2\\ -c_3s_1+s_3s_2c_1 & c_3c_1+s_3s_2s_1 & s_3c_2\\ s_3s_1+c_3s_2c_1 & -s_3c_1+c_3s_2s_1 & c_3c_2\end{bmatrix}$$

Outputs:
- $\mathbf{R}_{321}(\theta_1,\theta_2,\theta_3) \in \mathbb{R}^{3\times3}$ - rotation matrix for 3-2-1 rotation sequence

[1] In this reference, $\theta_1 = \psi$, $\theta_2 = \theta$, and $\theta_3 = \phi$.
[2] In this reference, $\theta_1 = \phi$, $\theta_2 = \theta$, and $\theta_3 = \psi$.

3-1-3 Rotation sequence

The 3-1-3 ($3 \to 1 \to 3$) Rotation sequence consists of the following steps:
1. rotation of θ_1 about x_3 (3rd axis)
2. rotation of θ_2 about x_1' (1st axis)
3. rotation of θ_3 about x_3'' (3rd axis)

The rotation matrix for the 3-1-3 sequence is [3], [10, pp. 763-764][3]

$$\begin{aligned}\mathbf{R}_{313}(\theta_1,\theta_2,\theta_3) &= \mathbf{R}_3(\theta_3)\mathbf{R}_1(\theta_2)\mathbf{R}_3(\theta_1)\\ &= \begin{bmatrix}\cos\theta_3 & \sin\theta_3 & 0\\ -\sin\theta_3 & \cos\theta_3 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta_2 & \sin\theta_2\\ 0 & -\sin\theta_2 & \cos\theta_2\end{bmatrix}\begin{bmatrix}\cos\theta_1 & \sin\theta_1 & 0\\ -\sin\theta_1 & \cos\theta_1 & 0\\ 0 & 0 & 1\end{bmatrix}\end{aligned}$$

$$\boxed{\mathbf{R}_{313}(\theta_1,\theta_2,\theta_3) = \begin{bmatrix}\cos\theta_3\cos\theta_1 - \sin\theta_3\cos\theta_2\sin\theta_1 & \cos\theta_3\sin\theta_1 + \sin\theta_3\cos\theta_2\cos\theta_1 & \sin\theta_3\sin\theta_2\\ -\sin\theta_3\cos\theta_1 - \cos\theta_3\cos\theta_2\sin\theta_1 & -\sin\theta_3\sin\theta_1 + \cos\theta_3\cos\theta_2\cos\theta_1 & \cos\theta_3\sin\theta_2\\ \sin\theta_2\sin\theta_1 & -\sin\theta_2\cos\theta_1 & \cos\theta_2\end{bmatrix}} \tag{1.45}$$

We formalize Eq. (1.45) as Algorithm 5 below. Note that in its implementation, we precompute the trigonometric functions *before* populating the matrix to decrease the computational cost.

Algorithm 5: `rot313`
Rotation matrix for the 3-1-3 rotation sequence.

Inputs:
- $\theta_1 \in \mathbb{R}$ - angle for first rotation (about 3rd axis) [rad]
- $\theta_2 \in \mathbb{R}$ - angle for second rotation (about 1st axis) [rad]
- $\theta_3 \in \mathbb{R}$ - angle for third rotation (about 3rd axis) [rad]

Procedure:
1. Precompute the trigonometric functions.

$$\begin{aligned}s_1 &= \sin\theta_1\\ c_1 &= \cos\theta_1\\ s_2 &= \sin\theta_2\\ c_2 &= \cos\theta_2\\ s_3 &= \sin\theta_3\\ c_3 &= \cos\theta_3\end{aligned}$$

2. Construct the rotation matrix.

$$\mathbf{R}_{313}(\theta_1,\theta_2,\theta_3) = \begin{bmatrix}c_3c_1 - s_3c_2s_1 & c_3s_1 + s_3c_2c_1 & s_3s_2\\ -s_3c_1 - c_3c_2s_1 & -s_3s_1 + c_3c_2c_1 & c_3s_2\\ s_2s_1 & -s_2c_1 & c_2\end{bmatrix}$$

Outputs:
- $\mathbf{R}_{313}(\theta_1,\theta_2,\theta_3) \in \mathbb{R}^{3\times 3}$ - rotation matrix for 3-1-3 rotation sequence

[3] In this reference, $\theta_1 = \phi$, $\theta_2 = \theta$, and $\theta_3 = \psi$.

1.8.5 Properties of Sequential Rotation Matrices

A sequential rotation matrix is defined as

$$\mathbf{R}_{ijk}(\theta_1, \theta_2, \theta_3) = \mathbf{R}_k(\theta_3)\mathbf{R}_j(\theta_2)\mathbf{R}_i(\theta_1)$$

$\mathbf{R}_i(\theta_1)$, $\mathbf{R}_j(\theta_2)$, and $\mathbf{R}_k(\theta_3)$ are all elementary rotation matrices, and in Section 1.8.3 we showed that elementary rotation matrices are orthogonal matrices. Additionally, from Section 1.7.2, we know that taking the product of orthogonal matrices results in an orthogonal matrix. Thus, we know that

$$\boxed{\text{Sequential rotation matrices, } \mathbf{R}_{ijk}(\theta_1, \theta_2, \theta_3)\text{, are orthogonal matrices.}}$$

Consider taking the transpose of $\mathbf{R}_{ijk}(\theta_1, \theta_2, \theta_3)$.

$$\begin{aligned}\mathbf{R}_{ijk}(\theta_1, \theta_2, \theta_3)^T &= [\mathbf{R}_k(\theta_3)\mathbf{R}_j(\theta_2)\mathbf{R}_i(\theta_1)]^T = \mathbf{R}_i(\theta_1)^T\mathbf{R}_j(\theta_2)^T\mathbf{R}_k(\theta_3)^T = \mathbf{R}_i(-\theta_1)\mathbf{R}_j(-\theta_2)\mathbf{R}_k(-\theta_3) \\ &= \mathbf{R}_{kji}(-\theta_3, -\theta_2, -\theta_1)\end{aligned}$$

Since sequential rotation matrices are orthogonal matrices, their transpose is also equal to their inverse. Thus,

$$\boxed{\mathbf{R}_{ijk}(\theta_1, \theta_2, \theta_3)^{-1} = \mathbf{R}_{ijk}(\theta_1, \theta_2, \theta_3)^T = \mathbf{R}_{kji}(-\theta_3, -\theta_2, -\theta_1) = \mathbf{R}_i(-\theta_1)\mathbf{R}_j(-\theta_2)\mathbf{R}_k(-\theta_3)} \tag{1.46}$$

PART II

Kinematics

PART III

Kinetics

PART IV

Appendices

A

Test Cases

A.1 Rotation Test Cases

A.1.1 Elementary Rotations

All rotation test cases should be tested to 15 digits of accuracy.

θ [rad]	$\mathbf{R}_1(\theta)$ `rot1`	$\mathbf{R}_2(\theta)$ `rot2`	$\mathbf{R}_3(\theta)$ `rot3`
0	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$
$\pi/4$	$\begin{bmatrix}1&0&0\\0&\sqrt{2}/2&\sqrt{2}/2\\0&-\sqrt{2}/2&\sqrt{2}/2\end{bmatrix}$	$\begin{bmatrix}\sqrt{2}/2&0&-\sqrt{2}/2\\0&1&0\\\sqrt{2}/2&0&\sqrt{2}/2\end{bmatrix}$	$\begin{bmatrix}\sqrt{2}/2&\sqrt{2}/2&0\\-\sqrt{2}/2&\sqrt{2}/2&0\\0&0&1\end{bmatrix}$
$\pi/2$	$\begin{bmatrix}1&0&0\\0&0&1\\0&-1&0\end{bmatrix}$	$\begin{bmatrix}0&0&-1\\0&1&0\\1&0&0\end{bmatrix}$	$\begin{bmatrix}0&1&0\\-1&0&0\\0&0&1\end{bmatrix}$
$3\pi/4$	$\begin{bmatrix}1&0&0\\0&-\sqrt{2}/2&\sqrt{2}/2\\0&-\sqrt{2}/2&-\sqrt{2}/2\end{bmatrix}$	$\begin{bmatrix}-\sqrt{2}/2&0&-\sqrt{2}/2\\0&1&0\\\sqrt{2}/2&0&-\sqrt{2}/2\end{bmatrix}$	$\begin{bmatrix}-\sqrt{2}/2&\sqrt{2}/2&0\\-\sqrt{2}/2&-\sqrt{2}/2&0\\0&0&1\end{bmatrix}$
π	$\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}$	$\begin{bmatrix}-1&0&0\\0&1&0\\0&0&-1\end{bmatrix}$	$\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix}$
$5\pi/4$	$\begin{bmatrix}1&0&0\\0&-\sqrt{2}/2&-\sqrt{2}/2\\0&\sqrt{2}/2&-\sqrt{2}/2\end{bmatrix}$	$\begin{bmatrix}-\sqrt{2}/2&0&\sqrt{2}/2\\0&1&0\\-\sqrt{2}/2&0&-\sqrt{2}/2\end{bmatrix}$	$\begin{bmatrix}-\sqrt{2}/2&-\sqrt{2}/2&0\\\sqrt{2}/2&-\sqrt{2}/2&0\\0&0&1\end{bmatrix}$

$3\pi/2$	$\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{bmatrix}$	$\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ -1 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$
$7\pi/4$	$\begin{bmatrix}1 & 0 & 0\\ 0 & \sqrt{2}/2 & -\sqrt{2}/2\\ 0 & \sqrt{2}/2 & \sqrt{2}/2\end{bmatrix}$	$\begin{bmatrix}\sqrt{2}/2 & 0 & \sqrt{2}/2\\ 0 & 1 & 0\\ -\sqrt{2}/2 & 0 & \sqrt{2}/2\end{bmatrix}$	$\begin{bmatrix}\sqrt{2}/2 & -\sqrt{2}/2 & 0\\ \sqrt{2}/2 & \sqrt{2}/2 & 0\\ 0 & 0 & 1\end{bmatrix}$
2π	$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$	$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$	$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

A.1.2 Sequential Rotations

3-2-1 Rotation Sequence (`rot321`)

"Forward" Test:

$$\theta_1 = 30\left(\frac{\pi}{180}\right)$$
$$\theta_2 = -40\left(\frac{\pi}{180}\right)$$
$$\theta_3 = 50\left(\frac{\pi}{180}\right)$$

$\mathtt{rot321}(\theta_1, \theta_2, \theta_3)$ should equal $\mathtt{rot1}(\theta_3)\mathtt{rot2}(\theta_2)\mathtt{rot3}(\theta_1)$

"Reverse" Test:

$$\theta_1 = 30\left(\frac{\pi}{180}\right)$$
$$\theta_2 = -40\left(\frac{\pi}{180}\right)$$
$$\theta_3 = 50\left(\frac{\pi}{180}\right)$$

$\mathtt{rot321}(\theta_1, \theta_2, \theta_3)^T$ should equal $\mathtt{rot3}(-\theta_1)\mathtt{rot2}(-\theta_2)\mathtt{rot3}(-\theta_3)$

3-2-1 Rotation Sequence (`rot321`)

"Forward" Test:

$$\theta_1 = 30\left(\frac{\pi}{180}\right)$$
$$\theta_2 = -40\left(\frac{\pi}{180}\right)$$
$$\theta_3 = 50\left(\frac{\pi}{180}\right)$$

$\mathtt{rot313}(\theta_1, \theta_2, \theta_3)$ should equal $\mathtt{rot3}(\theta_3)\mathtt{rot1}(\theta_2)\mathtt{rot3}(\theta_1)$

"Reverse" Test:

$\theta_1 = 30\left(\frac{\pi}{180}\right)$
$\theta_2 = -40\left(\frac{\pi}{180}\right)$
$\theta_3 = 50\left(\frac{\pi}{180}\right)$
$\texttt{rot313}(\theta_1, \theta_2, \theta_3)^T$ should equal $\texttt{rot3}(-\theta_1)\texttt{rot1}(-\theta_2)\texttt{rot3}(-\theta_3)$

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